Double Angle and Half Angle Identities

Written below are the double angles for sine, cosine, and tangent.

$$\sin 2x = 2\sin x\cos x \qquad \cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases} \qquad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Note there are three identities for cos 2x. If you forget the other two, just use the first and convert to all cosines. Then you will get the second identity for cos 2x. If you convert to all sines, then you get the third identity for cos 2x.

Now we illustrate some examples involving the use of the double angle identities.

Example: Use a double angle identity to find the exact value of $\sin\left(\frac{8\pi}{3}\right)$.

→ The angle is actually twice another angle. So set the angle equal to 2x to find x.

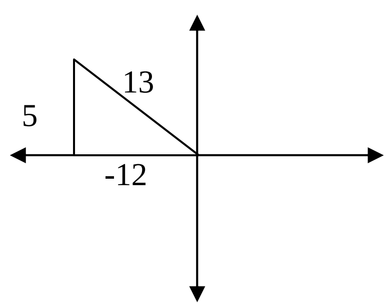
$$2x = \frac{8\pi}{3} \rightarrow x = \frac{4\pi}{3}$$

→ Now using the identity for sin 2x, we have

$$\begin{aligned} \sin\left(\frac{8\pi}{3}\right) &= \sin\left(2\cdot\frac{4\pi}{3}\right) \\ &= 2\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right) \\ &= 2\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Example: Given $\cos x = -\frac{12}{13}$, π/2 < x < π, find values for sin 2x, cos 2x, and tan 2x.

→ First draw and label a triangle in the given quadrant.



→ Now use the double angle identities to find the necessary values.

$$\sin 2x = 2\sin x\cos x = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right) = -\frac{120}{169}$$

$$\cos 2x = 2\cos^2 x - 1 = 2\left(-\frac{12}{13}\right)^2 - 1 = 2\left(\frac{144}{169}\right) - 1 = \frac{288}{169} - \frac{169}{169} = \frac{119}{169}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2} = \frac{-\frac{5}{6}}{1-\frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{144}{144}-\frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{119}{144}} = -\frac{5}{6}\cdot\frac{144}{119} = -\frac{120}{119}$$

You may be asked to find csc 2x, sec 2x, and cot 2x. All you have to do is find either sin 2x, cos 2x, or tan 2x and flip the answer.

Next we look at the half-angle identities for sine, cosine, and tangent.

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x}$$

Example: Use a half-angle identity to find the exact value of $\cos\left(\frac{7\pi}{12}\right)$.

→ The angle is actually half of another angle. So set the angle equal to x/2 to find x.

$$\frac{x}{2} = \frac{7\pi}{12} \rightarrow x = \frac{7\pi}{6}$$

→ Now using the identity for cos (x/2), we have

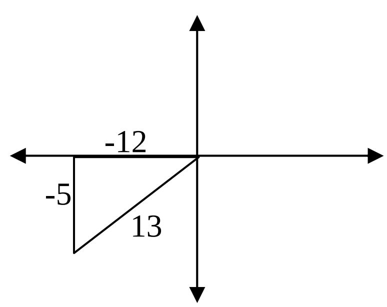
$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\frac{7\pi}{6}}{2}\right) = -\sqrt{\frac{1-\cos\frac{7\pi}{6}}{2}} = -\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = -\frac{\sqrt{2+\sqrt{3}}}{2}$$

Note we had to chose the negative sign for the answer because the angle lies in Quadrant II. Also the cosine of any Quadrant II angle is always negative.

Example: Given $\cos x = -\frac{12}{13}$, π < x < 3π/2, find values for sin (x/2), cos (x/2), and tan (x/2).

→ First draw and label a triangle in the given quadrant.



→ Now use the half-angle identities to find the necessary values.

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}} = -\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}} = -\sqrt{\frac{1+\frac{12}{13}}{2}} = -\sqrt{\frac{\frac{13}{13}+\frac{12}{13}}{2}} = -\sqrt{\frac{\frac{25}{13}}{2}}$$

$$= -\sqrt{\frac{25}{26}} = -\frac{5}{\sqrt{26}} = -\frac{5\sqrt{26}}{26}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}} = -\sqrt{\frac{1+\left(-\frac{12}{13}\right)}{2}} = -\sqrt{\frac{1-\frac{12}{13}}{2}} = -\sqrt{\frac{\frac{13}{13}-\frac{12}{13}}{2}} = -\sqrt{\frac{\frac{1}{13}}{2}}$$

$$= -\sqrt{\frac{1}{26}} = -\frac{1}{\sqrt{26}} = -\frac{\sqrt{26}}{26}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1-\left(-\frac{12}{13}\right)}{1+\left(-\frac{12}{13}\right)} = \frac{1+\frac{12}{13}}{1-\frac{12}{13}} = \frac{1+\frac{12}{13}}{1-\frac{12}{13}} \cdot \frac{\frac{13}{1}}{\frac{13}{1}} = \frac{13+12}{13-12} = \frac{25}{1} = 25$$

Similar to the double angle identities, you may be asked to find csc (x/2), sec (x/2), and cot (x/2). All you have to do is find either sin (x/2), cos (x/2), or tan (x/2) and flip the answer.